On the Accuracy of Firefly Synchronization with Delays

(Invited Paper)

Alexander Tyrrell Gunther Auer
DOCOMO Euro-Labs
80687 Munich, Germany
Email: lastname@docomolab-euro.com

Christian Bettstetter
University of Klagenfurt, Mobile Systems Group
9020 Klagenfurt, Austria
Email: Firstname.Lastname@uni-klu.ac.at

***Abstract*—Emergent synchronization in populations of fireflies is commonly described by the mathematical model of pulse-coupled oscillators (PCOs). This paper studies the achieved synchronization accuracy in the presence of coupling delays between PCOs. For a three node network, accuracy bounds in the stable state are derived. This case study proves useful when looking at meshed networks, where nodes may not be directly connected with all others. While the network topology impacts the achieved accuracy of PCO synchronization, simulations reveal that even for non-neighboring nodes the timing misalignment rarely exceeds twice the direct coupling delay.**

I. Introduction

Phenomena of emergent synchronization are ubiquitous in nature and are an inherent property common to many systems, which have intringued scientists since centuries. In the 17 th century, the Dutch scientist Huygens observed that two pendulum hanging from a common wooden bead naturally synchronize [1]. Other examples of emergent synchronization include heart cells [2], routing packets in telecommunication networks [3], and chaotic systems [4]. References [1, 5] provide comprehensive overviews of emergent synchronization phenomena.

Synchronization of connected entities are mathematically described as *coupled oscillators*: each entity naturally oscillates and influences others. In particular, entities interacting through discrete pulses are described by the theory of *pulse-coupled oscillators* (PCOs). This model describes systems such as the flashing of fireflies [6], the formation of earthquakes [7], and interacting neurons [8]. Mirollo and Strogatz [9] derived a mathematical model for synchronization in populations of PCOs. Under certain coupling conditions, it was proven that, for an arbitrary number of entities and independent of the initial conditions, the network always synchronizes [9].

The mathematical model of PCOs provides simple rules leading to synchronization, and has been applied to different fields, integrating different constraints. Applied to wireless networks, the emergent property of PCO synchronization enables nodes to align their internal timing reference in a distributed manner, starting from any initial misalignment. Various implementations and adaptations to wireless networks have been considered, and some include:

- utilizing the characteristic pulse of Ultra Wide Band (UWB) radio to imitate PCO synchronization [10];
- considering long synchronization sequences instead of pulses [11];
- placing the synchronization unit on the MAC layer, and performing synchronization through the exchange of low-level timestamps [12];

Delays are often neglected when studying the emergence of synchronization among PCOs [8, 9]. In [13] it was shown that delays limit the attainable accuracy in the stable synchronized state. For a network of two nodes an accuracy bound was derived. Furthermore, computer simulations suggested that these bounds are also valid for fully-meshed networks, i.e. networks where all nodes are directly connected [13].

In the present paper the work of [13] is extended to meshed networks, where nodes are not necessarily able to communicate directly with all others. In meshed networks PCO interactions over multiple hops influence the attainable accuracy. We elaborate the influence of the network topology on the accuracy bounds. For network topologies of three nodes, we derive conditions where the system remains in a stable state and establish the corresponding accuracy bounds. We demonstrate through simulations that the findings for the three node topologies are generally valid in meshed networks of arbitrary size.

The remainder of this article is structured as follows. Section II reviews the synchronization model for populations of PCOs. The impact of coupling delays on PCO synchronization is studied in Section III. Networks with two and three nodes are analyzed and serve as a basis to understand the achieved accuracy in meshed topologies with multiple nodes.

II. Synchronization of Pulse-Coupled Oscillators

Many natural system entities, such as a firefly emitting light pulses, a beating human heart, or a pendulum clock, display a common feature: they naturally oscillate. Their rhythm is determined by the properties of the system, and an internal source of energy compensates the dissipation. Such oscillators are thus autonomous, and classified as self-sustained oscillators within the class of nonlinear models.

In the following, each oscillator i is described by a phase function $\phi_i(t)$, $i \in \{1, \ldots, N\}$ where N is the number of

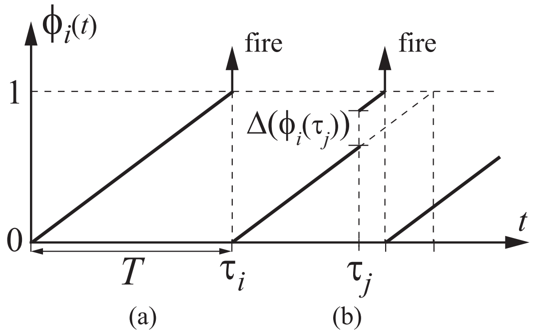


Fig. 1. (a) Uncoupled PCO phase function and (b) Phase increment upon reception of a pulse.

oscillators. This function evolves linearly over time from 0 and 1 with natural period T:

$$\frac{\mathrm{d}\phi_i(t)}{\mathrm{d}t} = \frac{1}{T} \; . \tag{1}$$

Whenever $\phi_i(t){=}1$ at reference instant $t{=}\tau_i$, the PCO is said to *fire*: it transmits a pulse and resets its phase to 0. Then $\phi_i(t)$ increases again linearly, and so on. We consider that all nodes have the same dynamics, i.e. clock jitter is considered negligible. The phase $\phi_i(t)$ can be seen as an internal counter that dictates the emission of pulses at instants $t{=}\tau_i$. Fig. 1(a) plots the evolution of the phase function during one period when the oscillator is isolated.

Received pulses implicitly provide information on the timing of surrounding oscillators, and are used to update internal phase functions. When node j fires at instant τ_j, receiving node i instantly increases its phase by a value Δ that depends on its current value $\phi_i(\tau_j)$:

$$\phi_i(\tau_j) \rightarrow \phi_i(\tau_j) + \Delta\left(\phi_i\left(\tau_j\right)\right) \; . \tag{2}$$

Fig. 1(b) plots the time evolution of the phase when receiving a pulse at $t{=}\tau_j$.

The phase increment $\Delta(\phi_i)$ is determined by the phase response curve (PRC), which was chosen to be linear in [9]:

$$\phi_i(\tau_j) + \Delta(\phi_i(\tau_j)) = \min\left(\alpha \cdot \phi_i(\tau_j) + \beta, 1\right) \tag{3}$$

where α and β determine the coupling between oscillators.

Provided that $\alpha{>}1$ and $0{<}\beta{<}1$, a network of N identical oscillators coupled all-to-all are always able to synchronize, so that all PCOs agree on a common reference instant, independent of the initial timing misalignments [9]. By appropriate choice of the coupling parameters α and β, in-phase synchronization, i.e. all nodes firing at the same instant, emerges within a few periods. In [14] the proof was extended to meshed networks.

Absorption: A key to understanding PCO synchronization is the absorption process. According to (3) all nodes whose phase is in the interval $[\phi_\ell, 1]$ increment their phase to 1 upon reception of a pulse and coalesce with the firing node, i.e. the received pulse forces them to fire instantly. The phase ϕ_ℓ is defined as the absorption limit and is equal to:

$$\phi_\ell = \frac{1-\beta}{\alpha} \; . \tag{4}$$

The absorption process repeats itself each time a node fires and if the phase value of some nodes is in the interval $[\phi_\ell, 1]$. The first firing node thus absorbs nodes in the absorption interval, and they resultantly form a cluster of nodes firing simultaneously.

For a population of oscillators, Mirollo and Strogatz show that over time fewer and fewer groups form through absorptions [9]. The dimension of the system is reduced each time an absorption occurs, and after a transient period where nodes cluster through absorptions, the system eventually attains a synchronized state of one cluster firing simultaneously.

III. Impact of Delays on the Synchronization of Pulse-Coupled Oscillators

A critical assumption in the PCO model presented in the previous section is that transmitted pulses *instantly* influence the phase of receiving nodes. In many systems delays are unavoidable. For example, when applying the PCO model to wireless networks, propagation delays between nodes need to be taken into account. Delays impact the stability as well as the attainable accuracy of PCO synchronization. This section elaborates on the stable states achieved by systems of two and three nodes in the presence of delays. These simple cases are then used to understand the behavior in larger networks, which are evaluated through simulations.

Refractory period: When delays are introduced, such as propagation delays, the coupling between two nodes i and j is delayed by ν_{ij}. In the presence of coupling delays a network of PCOs may become unstable, and the network is unable to synchronize [15].

To regain stability, a refractory period of duration T_{refr} after transmitting is introduced [13]. In refractory, i.e. when $\phi_i(t){<}\phi_{\text{refr}}$ with $\phi_{\text{refr}}{=}T_{\text{refr}}/T$, no phase increment is possible, so that received pulses are not acknowledged.

The duration of the refractory period needs to be at least twice the maximum propagation delay between two nodes, so that *echos* are not acknowledged: if node i fires at τ_i and forces node j to fire at $\tau_j{=}\tau_i{+}\nu_{ij}$, then the echo transmitted by node j is not acknowledged if node i is in refractory at $t{=}\tau_i{+}2\nu_{ij}$. This translates to the following stability condition [13]:

$$T_{\text{refr}} > \max_{i,j} 2\,\nu_{ij} \; . \tag{5}$$

Accuracy in the stable state: In the synchronized *stable state*, the difference between the reference instants of two nodes i and j is invariant over time and is defined as the achieved accuracy:

$$\epsilon_{ij} = |\tau_i - \tau_j| \; . \tag{6}$$

While the introduction of an appropriate refractory period T_{refr} ensures that a network of PCOs converges to a stable state, the attainable timing accuracy ϵ_{ij} is compromised.

A. Two Nodes

The accuracy limits for a network of $N{=}2$ nodes were derived in [13]. The accuracy is bounded by the interval of firing instants leading to a stable state as described in the following.

Suppose that the reference instants of two nodes i and j are aligned such that $\tau_j{>}\tau_i{+}\nu_{ij}$; then node i is the *forcing node* that imposes its *delayed* reference onto node j. After coupling, node j is pulled to the delayed timing of node i, $\tau_j{=}\tau_i{+}\nu_{ij}$ (as shown for nodes $i{=}1$ and $j{=}2$ in Fig. 3), as long as the pulse of node i falls within the absorption interval of node j, that is $\phi_j(\tau_i{+}\nu_{ij}) \in [\phi_j(\tau_j){-}\phi_\ell, \phi_j(\tau_j)]$. If $\tau_i{>}\tau_j{+}\nu_{ij}$, the roles are reversed, in the way that node j imposes its delayed timing onto node i, so that after coupling $\tau_i{=}\tau_j{+}\nu_{ij}$.

On the other hand, if the reference instant of node i is within the range

$$\tau_i \in [\tau_j{-}\nu_{ij}, \tau_j{+}\nu_{ij}] \ , \tag{7}$$

the pulses from node j fall into the refractory period of node i, and vice versa, and are thus not acknowledged. This corresponds to the stable state where the phases of both nodes are not adjusted.

Condition (7) states that the accuracy in a stable state (6) between directly connected nodes is bound by the coupling delay [13], that is $\epsilon_{ij} \in [0, \nu_{ij}]$.

B. Three Nodes

The analysis of [13] is extended to three nodes in the following. For a network of $N{=}3$ nodes, there exist two connected topologies, as shown in Fig. 2.

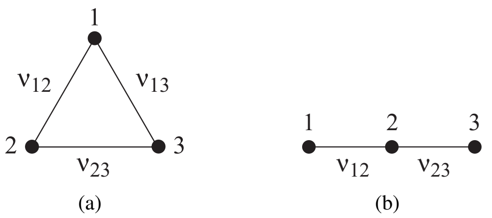


Fig. 2. Networks of three nodes: (a) Triangle topology, (b) Line topology.

a) Triangle Topology: In a triangle network (Fig. 2(a)), all nodes are directly connected. Suppose that node 1 is the forcing node that imposes its *delayed* timing onto the other nodes. This state is shown in Fig. 3: node 1 fires at instant $t{=}\tau_1$, which causes nodes 2 and 3 to increment their phases at instants $\tau_1{+}\nu_{12}$ and $\tau_1{+}\nu_{13}$ respectively. Assuming that their phase exceeds the absorption threshold, nodes 2 and 3 fire at instants $\tau_2{=}\tau_1{+}\nu_{12}$ and $\tau_3{=}\tau_1{+}\nu_{13}$, and subsequently enter refractory. No further phase increments occur, because the pulses from nodes 2 and 3 are received when nodes are in refractory (5). The $N{=}3$ triangle network thus reaches the stable state. In general, starting from a random initial condition, the system converges to a stable state where condition (7) is met for all pairs of connected nodes.

In this example, the achieved accuracies of node 1 relative to node 2 and 3 amounts to $\epsilon_{12}{=}\nu_{12}$ and $\epsilon_{13}{=}\nu_{13}$ respectively. Interestingly, the accuracy between nodes 2 and 3 is equal to the

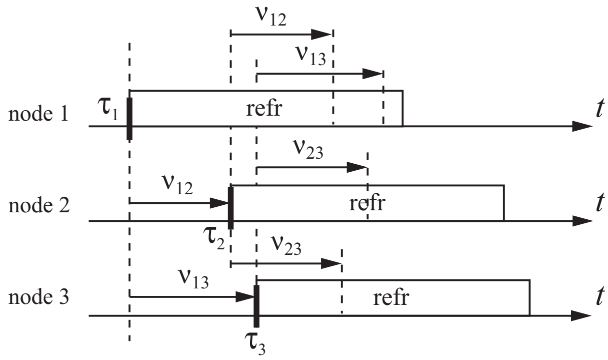


Fig. 3. Synchronization interactions for three nodes connected all-to-all.

difference in delays with forcing node 1, i.e. $\epsilon_{23}{=}|\nu_{12}{-}\nu_{13}|$. Thus the achieved accuracy does not depend on the direct delay ν_{23} but on the delay difference with the forcing node 1, as nodes 2 and 3 do not influence each other in the stable state.

b) Line Topology: When nodes form a line network (Fig. 2(b)), nodes 1 and 3 cannot communicate directly. Two stable states where one node forces its delayed timing on the others are distinguished. If node 2 fires first, it imposes its delayed timing onto the edge nodes 1 and 3. The resulting stable state corresponds to the triangle topology shown in Fig. 3, and, although edge nodes cannot communicate directly, the achieved accuracy between them is equal to $\epsilon_{13} = |\nu_{12} - \nu_{13}|$. If either edge node 1 or 3 fires first, it imposes its timing onto node 2, which in turn imposes its timing onto the other edge node. Due to the accuracy bound between two directly connected nodes (7), the resulting accuracy interval over two hops, between the edge nodes 1 and 3, is bounded by $\epsilon_{13} \in [0, \nu_{12} + \nu_{23}]$.

C. Multiple Nodes

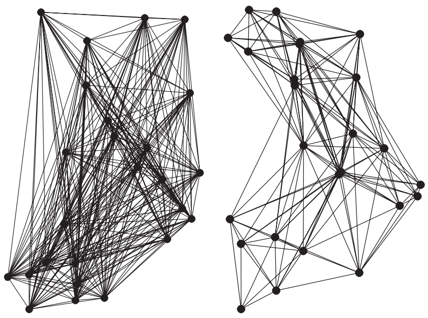
Provided sufficient refractory duration (5), the accuracy in the stable state between directly connected nodes i and j (7) is generally valid for a network of N PCOs with delays.

For a system of N nodes, two bounds on the achieved accuracy are easily derived by extending the three node cases in Section III-B. When all nodes can communicate directly with each other, the conditions to reach the stable state correspond to the triangle topology (Fig. 2(a)): the timing misalignment of node j is upper bounded by the coupling delays to the forcing node i, so that $\tau_j{\leq}\tau_i{+}\nu_{ij}$, $\forall\, j$. The accuracy is therefore bounded by the largest delay in the network, $\epsilon_{ij}{\leq}\max_{ij}\nu_{ij}$. A second bound is obtained by considering a line topology of $N{-}1$ hops. In this case, the worst accuracy is obtained when one of the edge nodes imposes its timing, and the rest of the chain follows it. Similar to three nodes forming a line (Fig. 2(b)), the worst accuracy between the two edge nodes is equal to the sum of delays along the line, $\epsilon_{1,N}{\leq}\sum_{i=2}^{N}\nu_{i-1,i}$.

Less trivial topologies are treated through simulations in the remainder of this section. The simple cases presented for two and three nodes and the accuracy bounds for fully-meshed and line topologies are instructive to understand the simulation results.

1) Meshed Network: The network topology is modeled as a *random geometric graph* $\mathcal{G}(N,r)$: N nodes forming the vertex set denoted by $\mathcal{V}$ are placed on a square area using a uniform random distribution, and nodes are connected if their distance is lower or equal than r. The set of links is denoted by $\mathcal{E}$, and two connected nodes are called *neighbors*. The set of neighbors of node i is defined as $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\}$. If all node pairs are connected by a link, the network is said to be *fully-meshed*.

A common measure to characterize topological properties of a network is its *algebraic connectivity* [16]. It conveniently summarizes the topology of a network and its degree of connectivity. The algebraic connectivity, denoted by κ, is the smallest non-zero eigenvalue of the Laplacian matrix $\mathbf{L}(\mathcal{G})$ [16]. This eigenvalue is strictly greater than 0 if and only if $\mathcal{G}$ is a connected graph [16]. Fig. 4 shows three examples of network topologies of $N{=}25$ nodes with different connectivities. For a fully-meshed network, $\kappa{=}N$. For a given N, the algebraic connectivity κ is varied by changing r, the maximum distance connecting two nodes.



(a) $\kappa/N{=}1$ (b) $\kappa/N{=}10^{-1}$ (c) $\kappa/N{=}10^{-2}$

Fig. 4. Examples of network topologies of networks of $N{=}25$ nodes for different normalized algebraic connectivities κ/N.

2) Scatter Plot: To evaluate the stable synchronized state, the achieved accuracy is measured after nodes have synchronized. Fig. 5 plots the achieved accuracy ϵ_{ij} in (6) as a function of the propagation delay ν_{ij}. Results in this figure differenciate between neighboring links, i.e. $(i,j) \in \mathcal{E}$, and nodes that are not able to communicate directly. Both propagation delays and achieved accuracy are normalized by ν_r the maximum delay between two connected nodes, i.e. $\nu_r = r/\mathrm{c}$ where c is the speed of light.

Results in Fig. 5 confirm that (7) is valid for networks with N nodes: the achieved accuracy between neighboring nodes is bounded by the propagation delay (7). For non-neighboring nodes, the accuracy is more scattered, and it sometimes exceeds the direct propagation delay. On the other hand, the achieved accuracy may also approach zero, even for large propagation delays. These large variations in achieved accuracy are explained by the analysis of the $N{=}3$ nodes network (see Fig. 3).

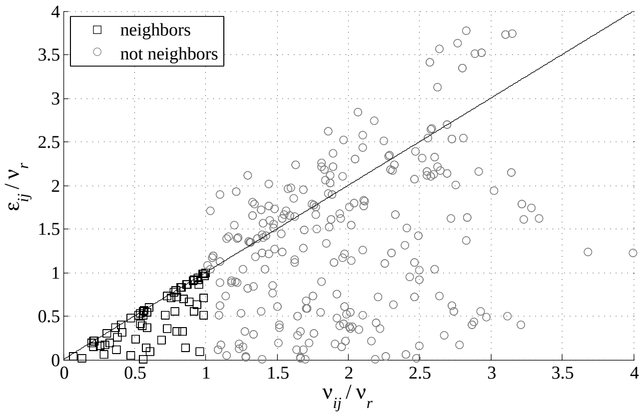


Fig. 5. Scatter plot of the achieved accuracy ϵ_{ij} as a function of the propagation delay ν_{ij} for a network of $N{=}25$ with normalized connectivity $\kappa/N{=}0.1$.

3) Normalized Accuracy: To assess how often the accuracy is below the propagation delay, we examine the distribution of the achieved synchronization accuracy. To this end, we define ρ_{ij} as the achieved accuracy normalized by the propagation delay:

$$\rho_{ij} = \frac{\epsilon_{ij}}{\nu_{ij}} \ . \tag{8}$$

a) Local and Global Accuracy: Fig. 6 plots the cumulative distribution function (cdf) of the normalized achieved accuracy ρ_{ij} for networks with a normalized algebraic connectivity of $\kappa/N{=}\ 10^{-2}$ (see Fig. 4(c)). The cdf is computed from 200 sets of initial conditions performed on 50 randomly generated networks of $N{=}25$ nodes.

Fig. 6 confirms results of the scatter plot. The synchronization accuracy among neighbors is below or equal to the propagation delay, i.e. $\rho_{ij} {\leq} 1$ in all cases. The cdf of ρ_{ij} also shows that 40% of neighboring links have an accuracy exactly equal to the propagation delay, observed by a jump in the cdf at $\rho_{ij}{=}1$. In 70% of cases, non-neighboring nodes achieve an accuracy below the direct propagation delay. The achieved accuracy among all non-neighboring nodes is always better than twice the propagation delay, i.e. $\rho_{ij}{<}2$.

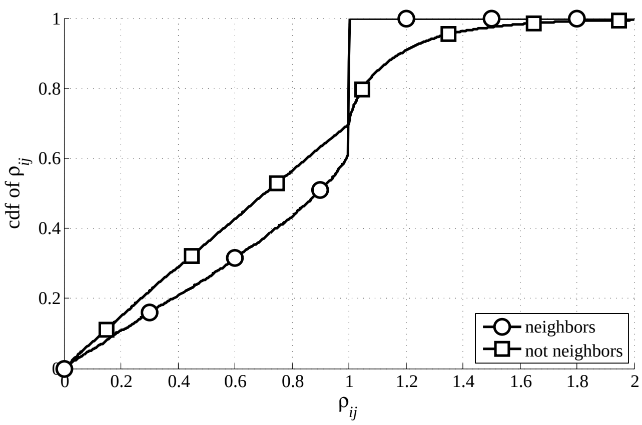


Fig. 6. Cdf of the achieved accuracy for networks of 25 nodes and an algebraic connectivity of $\kappa/N{=}10^{-2}$.

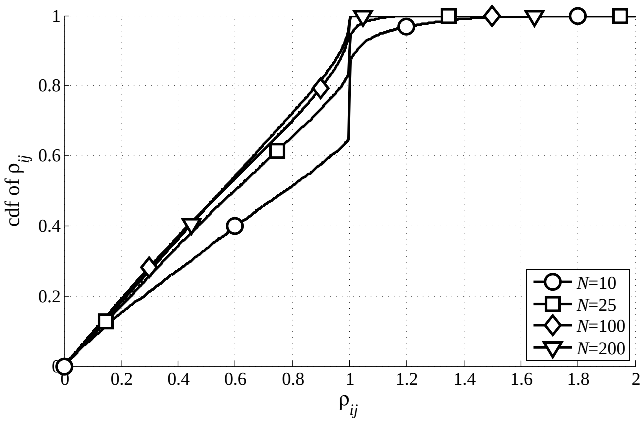


Fig. 7. Cdf of the normalized accuracy for networks of algebraic connectivity κ/N=0.1.

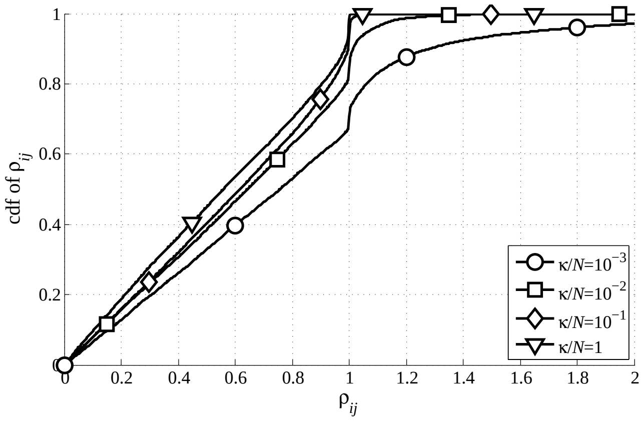


Fig. 8. Cdf of the normalized accuracy for networks of 50 nodes with various algebraic connectivities κ.

b) Influence of the Number of Nodes: Fig. 7 plots the achieved normalized accuracy as the number of nodes N varies. The normalized algebraic connectivity is constant and equal to $\kappa/N{=}10^{-1}$, and no distinction is made between neighboring and non-neighboring nodes.

Fig. 7 indicates that the normalized accuracy improves as the number of nodes in the network increases; for networks of over $N{=}100$ nodes, $\rho_{ij} \leq 1$ in all cases. This indicates that the probability that corner nodes impose their timing is diminishing as N increases. Rather middle nodes are likely to impose their timing, as for the three node case forming a line topology, as N increases, which results in improved accuracy. The abrupt jump at $\epsilon_{ij}{=}1$, due to achieved accuracies equal to the propagation delay, decreases as N increases.

c) Influence of the Connectivity: Fig. 8 plots the normalized accuracy for different algebraic connectivities and for a constant number of nodes $N{=}50$ In Fig. 8, for the fully-meshed case, i.e. $\kappa/N{=}1$, the normalized accuracy is always equal or below 1. This confirms the results that the achieved accuracy in a fully-meshed network is never larger than the propagation delay. As fewer direct links are present in the network and the connectivity diminishes, the accuracy also decreases, and the jump at $\rho_{ij}{=}1$ diminishes as κ decreases, because the number of neighbors is smaller. For a very low connectivity $\kappa/N{=}10^{-3}$, poor accuracies above twice the propagation delay occur, $\rho_{ij}{>}2$, although with low probability.

IV. Conclusion

This paper studied the synchronization of pulse-coupled oscillators (PCOs) in the presence of delays. For a network of three nodes conditions for convergence to the stable state were derived, which proved useful when examining larger networks. Further it was demonstrated that the network topology influences the achieved accuracy. Neighboring nodes are synchronized with an accuracy that is always equal or below the propagation delay, and in over 90% of cases, non-neighboring nodes are synchronized within twice the direct coupling delay.

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