

Efficient Binary Consensus in Randomized and Noisy Environments

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Abstract—In this article we investigate randomized binary majority consensus in networks with random topologies and noise. Using computer simulations, we show that asynchronous Simple Majority rule can reach $\simeq 100\%$ convergence rate being randomized by an update-biased random neighbor selection and a small fraction of errors. Next, we show that such gains are robust towards additive noise and topology randomization.

Keywords-distributed consensus; density classification; majority sorting; binary consensus; randomized consensus; wait-free consensus; self-organization;

I. INTRODUCTION

Consensus algorithms can be used in various distributed systems to perform tasks of mission planning [1], target detection and tracking [2] or database management [3].

In real-life networked systems algorithms with deterministic execution time can be beneficial. Such algorithms are called *wait-free* consensus algorithms: they are terminated after certain time T , whether overall agreement was achieved or not [4], [5]. Wait-free consensus is sensitive to disturbances and can decrease performance in noisy or asynchronous environments [6], [7].

However, disturbances not only hinder consensus [6] but can also promote it [8], [9], [10], [11]. Employment of random processes to promote consensus is called randomization [9]. It was shown that various types of randomization can increase fault tolerance [12] and convergence rate [8] of consensus algorithms.

Results reported in [8], [10], [11] motivated us to investigate the impact of randomization on binary majority consensus. In this article we study binary majority consensus in asynchronous networks. We focus on the impact of random neighbor selection and binary errors on Simple Majority (SM) consensus [8], [10]. Using computer simulations we show that asynchronous SM can outperform all known algorithms being randomized by biased neighbor selection and a small fraction of errors. Further, we show that gains

achieved due to such randomization are robust towards noise and topology randomization.

II. RELATED WORK

Performance of an algorithm for binary majority consensus is generally measured as convergence rate R — a fraction of initial network configurations that results in successful agreement (for details see section IV). Table I,

Table I
CONVERGENCE RATE OF ALGORITHMS FOR BINARY MAJORITY
CONSENSUS. REGULAR GRID OF $N = 149$ NODES.

| Rule | Year | Design | Ref. | $R, \%$ |
|--------------------|------|----------------------|------|---------------|
| SM | 1978 | human-designed | [6] | 81.6 |
| Davis | 1995 | human-designed | [13] | 81.8 |
| Koza | 1996 | genetic programming | [13] | 82.3 |
| Juille and Pollack | 1998 | co-evolution | [14] | 86.0 |
| Simple Majority | 2004 | randomized by errors | [8] | ≥ 85.0 |
| Traffic Majority | 2013 | randomized, biased | [15] | $\simeq 90.0$ |

partially borrowed from [13], presents the best R achieved by different methods during last several decades. Convergence rate is measured in a synchronous ordered network of $N = 149$ nodes to match original conditions [6]. The best result for deterministic (non-randomized) algorithm is shown by algorithm evolved by co-evolution is $R = 86\%$ [14].

Randomized algorithms [8], [15] can show high convergence rate. Simple Majority rule randomized by errors shows $R \geq 85\%$ [8] in random networks [16], while randomized Traffic Majority rule achieves $R \simeq 90\%$ [15] in ordered grids.

Lack of system synchrony generally decreases R [7] but SM consensus indicates higher R in asynchronous [8] and randomized environments [11].

Despite high R of SM in randomized environments, in ordered noiseless networks SM shows convergence rate below 1% (see Figure 4).

A challenging task therefore is to design a rule that is efficient in both randomized and ordered noisy networks.

Randomization is a technique used to increase R of consensus algorithms [9]. Although randomization can increase R , it can also affect the convergence time as termination becomes probabilistic [9]. This condition limits applicability of randomization to wait-free consensus, such as SM, where the algorithm is terminated after T steps, whether agreement was reached or not [17], [5].

In this paper we focus on wait-free SM consensus randomized by biased neighbor selection and a small fraction of errors. We show that such randomization can increase R of SM to $\simeq 100\%$. We also show that achieved gains in R are robust towards noise and topological randomization.

III. SETUP AND NOTATION

For network modeling we use Watts-Strogatz graph [16]. Network is initially modeled as the $2K$ -connected one-dimensional cellular automaton of N nodes, closed in a ring to avoid boundary effects (see Figure 1). Further, with rewiring probability P each link of the each node i is rewired to a random node j , $j \notin \{i - K, i + K\}$. I.e., at rewiring probability $P = 0$ a network is a regular $2K$ -connected ring. At $P = 0.5$ a network is a Small-World graph where $\simeq 50\%$ of the links are random (see Figure 2). At $P = 1$ a network is a pure random graph (Figure 3). Nodes that have links to the node i , sorted in ascending order, build nodes' i vector of neighbors N_i . Nodes $z \in \{N_i\}$, $z > i$ form a vector of right-side neighbors N_r . Nodes $z \in \{N_i\}$, $z < i$ form a vector of left-side neighbors N_l .

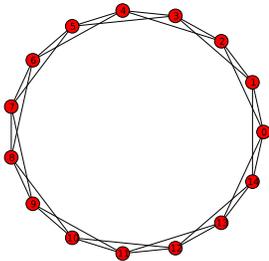


Figure 1. Network is a $2K$ — connected ring, at $P = 0$. $N = 15$, $K = 2$.

IV. DISTRIBUTED BINARY MAJORITY CONSENSUS

At the first time step $t = 0$ every node $i \in N$ is randomly assigned with a binary state $\sigma_i \in \{-1, 1\}$. A set of σ_i at $t = 0$ is called *Initial Configuration* and denoted as I . The sum of all σ_i at a time $t = 0$ is called the *Initial Density* and is denoted by ρ , $\rho \in \{-N, \dots, N\}$. At every time step $0 \leq t \leq T$ each node updates its state following a given consensus algorithm, based on its own state and state messages received from neighboring nodes.

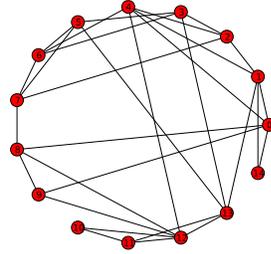


Figure 2. At $P = 0.5$, $\simeq 50\%$ of link are random, $N = 15$, $K = 2$.

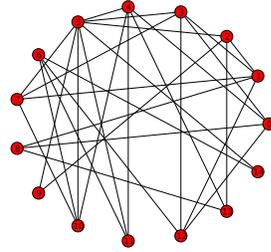


Figure 3. Network at $P = 1$, random graph of $N = 15$.

Within T time steps all nodes are expected to converge to a single state, corresponding to initial majority. I.e., the network is converged if there exists time $t \in \{0, \dots, T\}$, so that $\sum_{i=0}^{i=N} \sigma_i[t] = -N$ for $\rho < 0$, or $\sum_{i=0}^{i=N} \sigma_i[t] = N$ for $\rho > 0$. In our simulations we use $T = 2N$, originally proposed in [6].

A. Initial Configurations

We use test sets where each initial configuration is obtained by a series of N coin-flip operations, returning 1 and -1 with equal probability [8]. With such test sets SM shows lower R , than with test sets obtained by different methods [10].

B. Simple Majority Consensus

Simple Majority consensus [8] is defined as follows. Each node i on the network calculates its new state based on its own current state information and state information of its $2K$ nearest neighbors. I.e.:

$$\sigma_i[t + 1] = G \left(\sigma_i[t] + \sum_{j \in N_i} \sigma_{i,j}[t] \right). \quad (1)$$

The update function G is [8]:

$$G(x) = \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } x > 0 \end{cases}. \quad (2)$$

C. Random Majority Consensus

To investigate whether randomization by errors and random neighbor selection can be beneficial, we modify SM into Random Majority (RM). RM updates to the new state on the basis of its own state information and state information received from C neighbors randomly selected from the set of left-sided neighbors N_i :

$$\sigma_i[t+1] = G \left(\sigma_i[t] + \sum_{j=1}^C \sigma_{i,n_j}[t] \right), \quad (3)$$

where n_j is a neighbor randomly selected from N_i . To select n_j we sample from a discrete uniform distribution with repetition, i.e., the same neighbor can be selected more than once. State information $\sigma_{i,j}$ received by a node i is then substituted with erroneous with probability η :

$$\sigma_{i,j}[t] = \begin{cases} \sigma_j[t] & \text{with probability } (1 - \eta) \\ -\sigma_j[t] & \text{with probability } \eta \end{cases}. \quad (4)$$

We study Random Majority consensus with $C \in \{2, 4\}$ randomly selected neighbors and error probability $\eta \in \{0.05, 0.07, 0.08\}$. RM utilizes update-biased neighbor selection scheme, where node i selects neighbors from the set of left-side neighbors. Given a sequential update scheme (see Section IV-D), it allows the node to utilize the latest available state information of the neighboring nodes.

This rule differs from the Random Neighbor Majority consensus described in [11]. Random Majority, described here, employs biased random neighbor selection procedure and additional randomization by binary errors, in contrast to balanced neighbor selection in [11].

D. Update Mode

We use two common state update schemes: *synchronous* and *asynchronous* [7]. In synchronous mode, all nodes update their states simultaneously. In the asynchronous mode, nodes are updated sequentially, one after another, according to their indices. To update its state, each node uses the latest available states of its neighbors.

E. Additive Noise

We simulate additive noise at received messages by using the following transition:

$$\sigma_{i,j}[t] \rightarrow \sigma_{i,j}[t] + \phi_j. \quad (5)$$

We model noise as Additive White Gaussian Noise (AWGN), $\phi_j \sim \mathcal{N}(0, (\frac{A}{3})^2)$, $A \in [0, 1]$. Preceding studies [18], [19] also consider AWGN as the most common noise type in real environments.

V. PERFORMANCE STUDY

Performance is measured in terms of convergence rate R — a fraction of a 10,000 initial configurations that result in a successful agreement. We simulate over 30 sets combined of 10,000 I each and plot average values with 95% confidence intervals. We use the original $T = 2N$ [6], and simulate over networks with $N \in \{29, \dots, 160\}$.

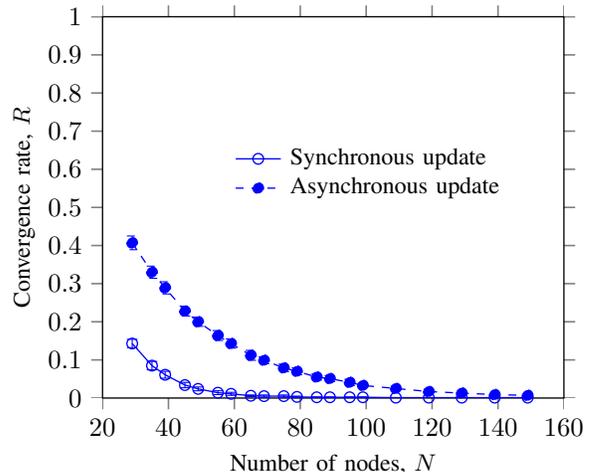


Figure 4. SM in ring lattices of $N \in \{29, \dots, 149\}$, $K = 3$.

A. Simple Majority in Ordered Networks

Figure 4 shows convergence rate of SM in ordered networks. These results resemble previously reported performance of SM in ordered networks [8], [11], showing that R is decreasing with larger N , and that lack of synchrony increases R of SM. Latter effects are studied in more detail in [11].

Further we focus on asynchronous networks, which generally inhibit consensus [7].

B. SM and RM in Ordered and Randomized Networks

Figure 5 shows convergence rate of asynchronous SM and RM over number of nodes N in ordered and randomized WS networks. It illustrates that in ordered networks RM outperforms SM and other algorithms for binary majority consensus reaching $R \simeq 100\%$. In random networks SM outperforms RM with $N \leq 40$. In random networks RM shows $R \simeq 68\%$, which is robust towards system growth, although lower than R of the best algorithms (see Table I).

C. Robustness Towards Additive Noise

Figure 6 shows influence of additive noise on asynchronous RM with $C = 2$ and $\eta \in \{0.05, 0.07, 0.08\}$ in ordered grids. It indicates that in ordered grids, RM with $C = 2$ can outperform all known algorithms with noise magnitudes $A \leq 0.2$ for $\eta = 0.05$, $A \leq 0.4$ for $\eta = 0.07$, $A \in [0.1, 0.5]$ for $\eta = 0.08$.

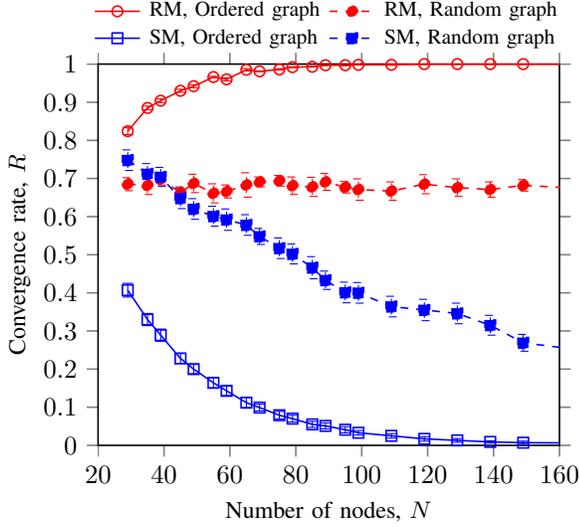


Figure 5. Asynchronous SM and RM in ordered and random networks. “Ordered graph” stands for $P = 0$, “Random graph” stands for $P = 1$. $N \in \{29, \dots, 160\}$, $K = 3$, $C = 4$, $\eta = 0.08$.

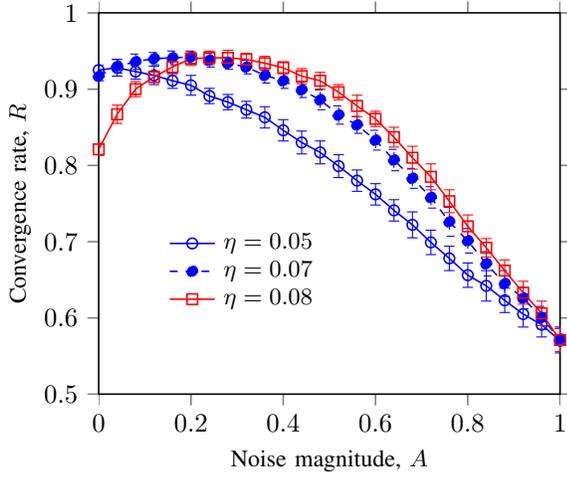


Figure 6. Asynchronous RM in ordered grids ($P = 0$) with additive noise. $N = 99$, $K = 3$, $C = 2$.

Figure 7 shows R of RM with $C = 4$ randomized by additive noise. It shows that with larger number of randomly selected neighbors RM increases R and robustness to noise. RM with $C = 4$ and $\eta \in \{0.07, 0.08\}$ shows $R \simeq 100\%$. To our knowledge this is the best result for distributed binary majority consensus in ordered grids. With growing noise magnitude R decreases to 90% (previous best R for randomized rule) at $A = 0.3$.

D. Robustness Towards Randomized Topology

Figures 8 and 9 show how topology randomization changes R of asynchronous RM with $C \in \{2, 4\}$ and $\eta \in \{0.05, 0.07, 0.08\}$ in noiseless networks. Figure 8

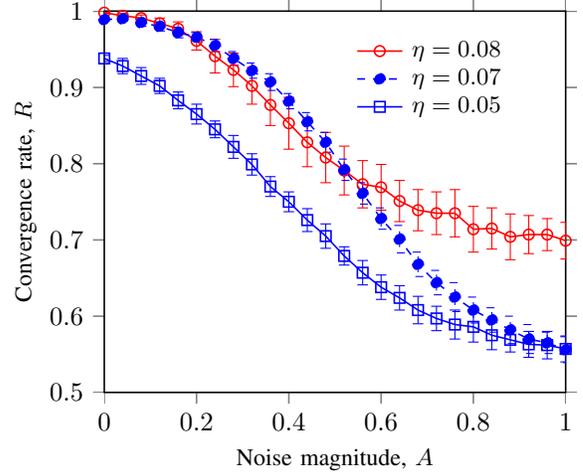


Figure 7. Asynchronous RM in ordered grids ($P = 0$) with additive noise. $N = 99$, $K = 3$, $C = 4$.

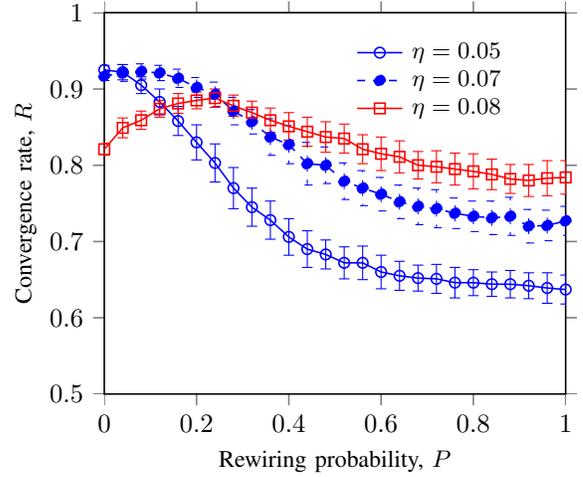


Figure 8. Asynchronous RM in noiseless networks ($A = 0$) with topology randomization. $N = 99$, $K = 3$, $C = 2$.

shows that RM with $C = 2$ and $\eta \in \{0.05, 0.07\}$ can reach $R \geq 90\%$ with topology randomization $P \leq 10\%$ and $P \leq 20\%$ respectively. It also indicates that with $\eta = 0.08$ RM shows $R \geq 82\%$ with $P \leq 0.6$.

Figure 9 indicates that increase of C from 2 to 4 increases R and promotes robustness towards topology randomization. Thus, maximum R is increased to $\simeq 100\%$ and robustness towards topology randomization is promoted to $P = 0.45$ at the convergence rate of $R \geq 90\%$.

Figures 6 — 9 show that employed randomization schemes provide different influence. Enforced errors on received information (η) significantly increase robustness towards noise and random topologies, while increase in the number of randomly selected neighbors (C) promotes growth of R . Combined, these two types of randomization can increase convergence rate of SM from $R \simeq 1\%$ (in the

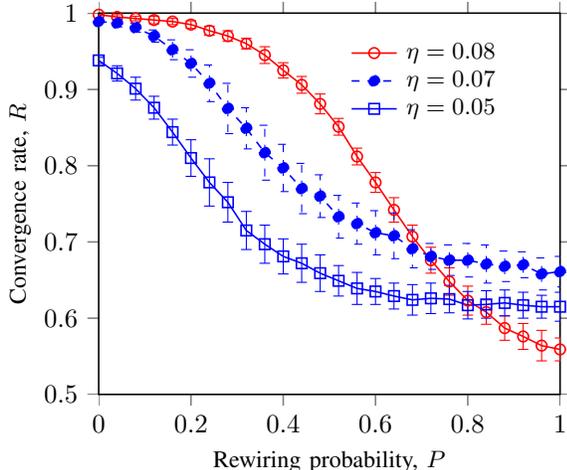


Figure 9. Asynchronous RM in noiseless networks ($A = 0$) with topology randomization. $N = 99, K = 3, C = 4$.

network of $N = 149$) to $R \simeq 100\%$. Similar effects were previously shown to promote SM consensus [8]. Land and Belew [20] show that consensus algorithms cannot reach $R = 100\%$, however, this restriction applies to deterministic algorithms. Overall, these results can be summarized as follows:

- RM consensus, randomized by errors and biased neighbor selection, shows R higher than any other algorithm for binary majority consensus in ordered grids;
- RM shows R higher than other algorithms with noise magnitudes $A \leq 0.35$ and topological randomization of $P \leq 0.45$;
- In topologically randomized environments RM shows R lower than Random Neighbor Majority (with noise) [11].

Random Majority consensus employs biased randomization by neighbor selection and randomization by errors enforced on received information. This embedded randomization promotes consensus in ordered grids and increases robustness towards noise and topology randomization. As mentioned earlier, Random Majority utilizes randomization scheme different from that described in [11]. Due to randomization it shows higher R than Random Neighbor Majority, described in [11], but lower robustness towards noise and randomized topology.

VI. CONVERGENCE ANALYSIS

Below we analyze the convergence dynamics of the RM. Figure 10 shows density evolution in the system with asynchronous RM over time. ρ is registered at each time step $t \in \{0, \dots, T\}$ for independently evolved 1.000 initial configurations. Density evolution shows that systems tend to converge to a correct majority within the time limits, although system exhibits a fraction of stochastic switching.

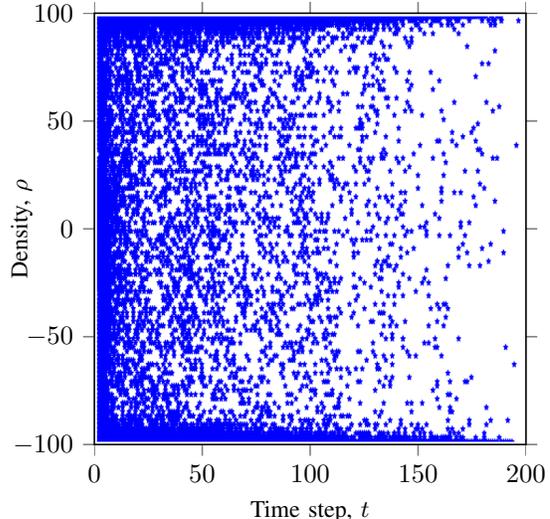


Figure 10. Density evolution over time, asynchronous RM in ordered networks ($P = 0$). $N = 99, K = 3, C = 4, \eta = 0.05$.

This explains the high R of RM but indicates that randomized binary consensus cannot guarantee convergence in a wait-free manner. This happens due to random dynamics that can disrupt the state of agreement.

VII. CONCLUSIONS

In this article we study binary majority consensus in asynchronous networks with randomized topologies and additive noise. Using computer simulations, we show that a small fraction of errors and random neighbor selection can increase convergence rate of SM consensus to $\simeq 100\%$. Convergence rate achieved by proposed Random Majority consensus ($\simeq 100\%$) exceeds that of all other published results for both ordered ($R = 86\%$ [14], $R \simeq 90\%$ [15]) and random ($R \geq 90\%$, [8]) networks. Gains, achieved due to such randomization, are robust towards topology randomization and noise. Convergence analysis shows that although systems converge to the correct agreement within a given time T , it does not guarantee stable agreement in a wait-free manner. This extends results previously reported in [10], [11].

ACKNOWLEDGMENT

This work was performed within the Erasmus Mundus Joint Doctorate in “Interactive and Cognitive Environments”, which is funded by the EACEA Agency of the EC under EMJD ICE FPA n 2010-0012. The work of A. Gogolev is supported by Lakeside Labs, Klagenfurt, with funding from the ERDF, KWF, and the state of Austria under the grant 20214/21530/32606.

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